1 Review of ODE

This first section will serve as a quick review of the pertinent material from *Math 266: Introduction to Ordinary Differential Equations*, which will be of significant importance for the rest of the course. More material will be reviewed as the course progresses.

1.1 Notation

This subsection serves to fix the notation.

Definition 1.1. *An ODE of the k-th order is the expression of the form*

$$\frac{\mathrm{d}^k x}{\mathrm{d}t^k} = f\left(t, x, \frac{\mathrm{d}x}{\mathrm{d}t}, \ldots, \frac{\mathrm{d}^{k-1}x}{\mathrm{d}t^{k-1}}\right). \tag{1.1}$$

Here I use t for the *independent variable* and x for the *dependent variable*. A function φ is called a *solution* to (1.1) on the interval $I = (a, b)$ if this function k times continuously differentiable on I (this is denoted usually as $\varphi \in \mathcal{C}^{(k)}(I; \mathbf{R})$, in words: Function φ with the domain I and range $\mathbf{R}$ belongs to the space of k times continuously differentiable functions) and after plugging this function into (1.1) turns this equation into the identity

$$\frac{\mathrm{d}^k \varphi}{\mathrm{d}t^k}(t) \equiv f\left(t, \varphi(t), \frac{\mathrm{d}\varphi}{\mathrm{d}t}(t), \ldots, \frac{\mathrm{d}^{k-1}\varphi}{\mathrm{d}t^{k-1}}(t)\right), \quad t \in I.$$

For example, for the equation of the radioactive decay

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\gamma x, \quad \gamma > 0,$$

the function

$$x(t) = Ce^{-\gamma t}$$

is a solution for any constant $C \in \mathbf{R}$ (check this) and for any interval $I \subseteq \mathbf{R}$.

The *order* of an ODE is the order of the highest derivative. Hence the first order ODE is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(t, x). \tag{1.2}$$

The solution formula to the equation of radioactive decay above actually gives infinitely many solutions, and this is true for most ODE: in a general situation the solution to an ODE depends on arbitrary constants, whose number coincides with the order of the equation.

Exercise 1. Can you solve the following ODE:

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \lambda x = 0,$$

where λ is a parameter? Note that the solution depends on the sign of λ. In any case your solution must depend on two arbitrary constants.

Math 480/680: *Applied Differential Equations* by Artem Novozhilov©
e-mail: artem.novozhilov@ndsu.edu. Fall 2017

In the context of mathematical models it is meaningless to have infinitely many solutions (think of a population growth: We cannot have infinitely many functions describing it). Therefore, we need some additional conditions to choose *the* solution that is of interest for us. This is done with the help of the *initial conditions*. The initial conditions for equation (1.1) take the form

$$\begin{aligned} x(t_0) &= x_0, \\ \frac{\mathrm{d}x}{\mathrm{d}t}(t_0) &= x_1, \\ &\vdots \\ \frac{\mathrm{d}^{k-1}x}{\mathrm{d}t^{k-1}}(t_0) &= x_{k-1}, \end{aligned} \tag{1.3}$$

where $x_0, \dots x_{k-1}$ are given numbers, and notation $\frac{\mathrm{d}^k x}{\mathrm{d}t^k}(t_0)$ means the k-th derivative of function x evaluated at the point t_0. Note that the number of the initial conditions is equal to the order of the ODE.

Definition 1.2. *ODE* (1.1) *plus the initial conditions* (1.3) *are called an initial value problem* (*abbreviated an IVP*) *or Cauchy's problem.*

For the equation of radioactive decay I hence need only one initial condition (the mass of my material at some time moment t_0). Check that if the initial condition $x(t_0) = x_0$, then the solution to the corresponding Cauchy problem is given by

$$x(t) = x_0 e^{-\gamma(t-t_0)}.$$

Any ODE of the form (1.1) can be written as a system of k first-order equations. Moreover, writing ODE as a system is theoretically the way how ODE should be treated. Hence I will usually write ODE as first order systems. To rewrite (1.1) as a system, I introduce new variables

$$x_1(t) = x(t),\, x_2(t) = \frac{\mathrm{d}x}{\mathrm{d}t}(t), \dots, x_k(t) = \frac{\mathrm{d}^{k-1}x}{\mathrm{d}t^{k-1}}(t).$$

Using these new variables I have

$$\begin{aligned} \frac{\mathrm{d}x_1}{\mathrm{d}t} &= x_2, \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= x_3, \\ &\vdots \\ \frac{\mathrm{d}x_k}{\mathrm{d}t} &= f(t, x_1, x_2, \dots, x_k). \end{aligned} \tag{1.4}$$

Note that the initial conditions for (1.4) become

$$x_1(t_0) = x_1, \dots, x_k(t_0) = x_k. \tag{1.5}$$

From now on for the first order derivatives I am going to use Newton's dot notation (which is widely used in the classical mechanics):

$$\dot{x} := \frac{\mathrm{d}x}{\mathrm{d}t}\,.$$

1.2 ODE as mathematical models

The usual process of *mathematical modeling* goes in several stages: First, we start with the situation at hands and formulate the main features of the considered system (physical, chemical, biological, etc), which we would like to retain in our mathematical model. At the same stage we disregard many unimportant for us details. After this first stage we formulate a mathematical model, which is built on our simplifying assumptions. If we have the model, we can forget about the original system and perform an analysis of the obtained mathematical problem. Finally, the solutions to this model should be interpreted in terms of the original system. Of course, the whole process is usually much more involved, but the outlined above line of reasoning can be found in many real world modeling situations. I will present a great number of examples in this course.

As a very basic example of the modeling approach, let me introduce the so-called *Malthus equation*. A very simple biological process of a population growth is considered. Let $N(t)$ denote the number of individuals in a given population (for concreteness you can think of a population of bacteria) at the time moment t. In this course the variable t will almost exclusively denote *time*. Now I calculate how the population number changes during a short time interval h. I have

$$N(t+h) = N(t) + bhN(t) - dhN(t).$$

Here I used the fact that the total population at the moment $t+h$ can be found as the total population at the moment t plus the number of individuals born during time period h minus the number of died individuals during time period h. b and d are per capita *birth* and *death rates* respectively (i.e., the numbers of births and deaths per one individual per time unit respectively). From the last equality I find

$$\frac{N(t+h) - N(t)}{h} = (b-d)N(t).$$

Next, I *postulate* the existence of the derivative

$$\frac{\mathrm{d}N}{\mathrm{d}t} = \lim_{h\to 0} \frac{N(t+h) - N(t)}{h},$$

assume for simplicity that both b and d are constant, and hence obtain an ordinary differential equation

$$\frac{\mathrm{d}N}{\mathrm{d}t} = (b-d)N,$$

which is usually called in the biological context the *Malthus equation* (I will come back to Malthus). Finally, I rewrite the equation in the form

$$\dot{N} = mN, \quad N(0) = N_0, \tag{1.6}$$

where $N(t)$ is the population size at the time moment t, m is the parameter of the model, $m = b - d$.

How did Malthus arrived at this mathematical model? Obviously, the process of the population growth or decline is very intricate, which is subject to many important factors, such as weather, temperature, diseases, religion and so on. Malthus stated his simplifying assumption that the population growth is "geometric," by this he meant that the population size increases in a geometric progression, which can be described as the relation $N(t+h) = wN(t)$, where w is the parameter of the geometric growth. In the terms of the continuous time, this means exactly the equation (1.6). So, he had a simplifying assumption, and his mathematical model — (1.6). Now I analyze the mathematical model, in this case I can simply solve it (see below):

$$N(t) = N_0 e^{mt}, \tag{1.7}$$

which predicts the *exponential growth* if $m > 0$. At the same time Malthus argued that the goods increase in the world *linearly*. Exponential population growth plus linear increase of food and similar things together mean, by Malthus, a catastrophe. Here is what Malthus wrote

> A man who is born into a world already possessed, if he cannot get subsistence from his parents on whom he has a just demand, and if the society do not want his labour, has no claim of right to the smallest portion of food, and, in fact, has no business to be where he is. At nature's mighty feast there is no vacant cover for him. She tells him to be gone, and will quickly execute her own orders, if he does not work upon the compassion of some of her guests. If these guests get up and make room for him, other intruders immediately appear demanding the same favour. The report of a provision for all that come, fills the hall with numerous claimants. The order and harmony of the feast is disturbed, the plenty that before reigned is changed into scarcity; and the happiness of the guests is destroyed by the spectacle of misery and dependence in every part of the hall, and by the clamorous importunity of those, who are justly enraged at not finding the provision which they had been taught to expect. The guests learn too late their error, in counter-acting those strict orders to all intruders, issued by the great mistress of the feast, who, wishing that all guests should have plenty, and knowing she could not provide for unlimited numbers, humanely refused to admit fresh comers when her table was already full.
>
> Thomas Robert Malthus (13 February 1766 — 23 December 1834)
> *An Essay on the Principle of Population*, Second edition
> (this quotation was removed from the text in the subsequent editions)

An obvious drawback of the Malthus equation is that it predicts an unlimited population growth to infinity, which is clearly unrealistic. The fact that no population can grow to infinity should be included in our mathematical models if we would like to consider predictions of the population size in the future times.

I can assume that the law of growth has the general form

$$\dot{N} = NF(N),$$

where F is some function, which has to be negative for sufficiently large values of N (do you see why it is important?). If this function is smooth enough, I can represent it with the help of the Taylor formula around $N = 0$:

$$F(N) = F(0) + \frac{F'(0)}{1!}N + \frac{F''(0)}{2!}N^2 + o(N^2).$$

Here the notation $f(N) = o(g(N))$ means that

$$\lim_{N\to 0} \frac{f(N)}{g(N)} = 0,$$

and I also assume that this term is negligible when $N \to \infty$.

Note that if in the Taylor formula I keep only the constant term, I obtain exactly the Malthus equation

$$\dot{N} = mN,$$

where $m = F(0)$. If I keep two terms, I obtain the equation

$$\dot{N} = NF(N) = N\big(F(0) + F'(0)N\big) = mN\left(1 - \frac{N}{K}\right),$$

where I used another parametrization (do you see how $F(0)$ and $F'(0)$ are connected to m and K?), which is the *logistic equation*, and the parameter K is the *carrying capacity.* Therefore, I presented a mechanistic argument in favor of the logistic equation as the simplest first order differential equation describing the population growth apart of the Malthus equation. I hope that at this point you can find, given $N(0) = N_0$, that the logistic equation has the solution

$$N(t) = \frac{N_0 K}{N_0 + (K - N_0)e^{-mt}} \to K, \quad t \to \infty.$$

We will see a lot of different mathematical models of biological systems in this course.

1.3 Solving ODE

Most of the time in your Math 266 course was devoted to finding analytical solutions to ODE. For example, here is how you can find the solution to the so-called *Malthus equation*, which is technically the same as the equation for the radioactive decay.

Example 1.3. Consider the Malthus equation that describes an exponential growth of a population of size $N(t)$ at the time moment t:

$$\dot{N} = mN, \quad N(t_0) = N_0,$$

where $m \in \mathbf{R}$ is the so-called Malthusian fitness, parameter of the problem. This equations is an example of a *separable* ODE of the form

$$\dot{x} = f_1(t) f_2(x),$$

when you can separate the variables. To solve the Malthus equation, I write

$$\dot{N} = mN \implies \int \frac{\mathrm{d}N}{N} = \int m\,\mathrm{d}t \implies \log|N| = mt + C \implies N(t) = Ce^{mt}.$$

Here I used log to denote the natural logarithm, and C is an arbitrary constant, which can be different from line to line. Using the initial condition, I find the soltion

$$N(t) = N_0 e^{m(t - t_0)}.$$

A word of caution is in order here, as the next example shows.

Example 1.4. Consider the following IVP:

$$\dot{x} = 4tx^2, \quad x(1) = 0.$$

The usual way the students solve this problem is as follows: They separate the variables

$$\frac{\mathrm{d}x}{x^2} = 4t \implies -\frac{1}{x} = 2t^2 + C \implies x = \frac{1}{C - 2t^2}\,.$$

Now they try to find the constant C such that

$$0 = \frac{1}{C-2}\,.$$

And here problems start, since there is no such finite C. Can you figure out what the right way to solve this problem?

Actually, I notice that while integrating this equation I divided by x^2. This expression turns into 0 for $x = 0$. This means that $x(t) = 0$ is a solution to our equation. Moreover, this solution satisfies the given initial condition. Therefore, the answer to this problem is

$$x(t) = 0.$$

Anyway, I will not need much of this technique in our course. However, it is useful to remember the basic methods of solving the first order ODE. It is also useful to know that *most* ODE cannot be solved in terms of elementary functions and finite number of integrals of them (the only large class of ODE that can be solved is the *linear ODE with constant coefficients*, and you studied this class of ODE extensively in Math 266). For example, this innocently looking equation

$$\dot{x} = x^2 - t$$

cannot be solved in terms of elementary function. Note that this does not mean that there is no solution, it means that the usual repertoire of the polynomial, logarithmic, exponential, and trigonometric functions, and four arithmetic operations and integrals are not enough to write down a solution to this equation (this deep result has the name *Liouville's theorem* and studied in the *Differentiable algebra* course). Therefore, we will need other means to solve ODE. What we will study most in this course is the *qualitative or geometric analysis* of ODE. This means that we will study the properties of solutions of ODE without actually having a formula for these solutions.

Another useful approach to solve ODE is the *numerical methods*. However, I will not discuss them in this course (look for a *Numerical analysis* course).

But even if one can solve an equation, it is not actually always clear what is the behavior of its solutions.

Example 1.5. Consider the equation

$$\dot{x} = \sin x.$$

We can integrate this equation (do this) and use the initial condition $x(t_0) = x_0$. But can we actually figure out the behavior of its solutions? What happens with the solution if $x(0) = \pi/4$ and $t \to \infty$?

1.4 Well-posed problems. Theorem of existence and uniqueness

I consider ODE as *mathematical models* of some, say, physical or biological systems, and as such they should possess some desirable properties. In particular, one such property is to be *well-posed* (do not think that if a problem is ill posed, it is impossible to consider it as a mathematical model; actually, a good deal of ill posed problems are of great interest in applications, but this does not belong to this course). The notion of a well posed problem was introduced by the French mathematician Jacques Salomon Hadamard.

A mathematical problem is well posed if

- Its solution exists.
- Its solution is unique.
- Its solution depends continuously on the initial data.

It turns our that the mathematical problems (1.1), (1.3) or (1.4), (1.5) are well posed if I require some mild technical conditions on f. Here I would like to provide the exact mathematical statements for the first order ODE.

Consider the following IVP for a first order ODE:

$$\dot{x} = f(t, x), \quad x(t_0) = x_0. \tag{1.8}$$

The following general result is usually discussed in Math 266 without proof.

Theorem 1.6. *Consider the IVP* (1.8) *and assume that function f is continuous in t and continuously differentiable in x for $(t, x) \in (a, b) \times (c, d)$ for some constants a, b, c, d, assume that $(t_0, x_0) \in (a, b) \times (c, d)$. Then there exists an $\varepsilon > 0$ such that the solution φ to* (1.8) *exists and unique for $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$.*

Note that Theorem 1.6 is *local*, i.e., it guarantees that the solution exists and unique on some in general smaller interval $(t_0 - \varepsilon, t_0 + \varepsilon) \subseteq (a, b)$. This is important because solutions to ODE can *blow up*, i.e., approach infinity for a finite t.

Example 1.7. Consider the ODE

$$\dot{x} = 1 + x^2 = f(t, x).$$

The right-hand side is a polynomial for any $(t, x) \in \mathbf{R} \times \mathbf{R}$. Its solution is given by (check this)

$$x(t) = \tan(t + C),$$

and hence for each fixed C is defined only on the interval $(-\pi/2 - C, \pi/2 - C)$ (you should sketch several solutions).

Here is an example how non-uniqueness can appear.

Example 1.8. Consider

$$\dot{x} = \sqrt{x}, \quad x(0) = x_0, \quad x \geq 0.$$

One solution is given, as can be found by the separation of variables, as

$$x(t) = (t + 2\sqrt{x_0})^2/4.$$

On the other hand, if $x_0 = 0$ then there is a solution which is identically zero for any t. Therefore through the point $(t_0, x_0) = (0, 0)$ more than one solution passes (actually infinitely many, can you find them all?).

To formulate the result on the dependence on the initial conditions consider, together with (1.8), the IVP

$$\dot{x} = f(t, x), \quad x(t_0) = x_1. \tag{1.9}$$

Theorem 1.9. *Let the IVP* (1.8) *and* (1.9) *satisfy the conditions of Theorem 1.6, and* $t \mapsto x_0(t)$ *and* $t \mapsto x_1(t)$ *be the solutions to* (1.8) *and* (1.9) *respectively at the same time* t. *Then*

$$|x_1(t) - x_0(t)| \leq |x_1 - x_0| e^{L|t-t_0|},$$

where L *is a constant that depends on* f.

Theorem 1.9 shows that the solution to a first order ODE depends *continuously* on the initial condition. This means that if the initial condition is known only approximately, I still can rely on the solution of the IVP as an approximation of the unknown solution, albeit for a sufficiently small time time interval $[t_0, T)$.

1.5 Geometric interpretation of the first order ODE

First order ODE

$$\dot{x} = f(t, x)$$

has a very useful and transparent geometric interpretation. Recall that the derivative of a function at a given point gives the slope of the tangent line to the graph of this function. Since at the right hand side of the first order ODE I have the value of the derivative $\dot{x}$ at the point (t, x), hence I know the slope at this particular point, and hence I also know the slope at any point at which f is defined. I can depict these slopes as small line segments at each point. All together these line segments form a *direction* (or *slope*) *field.* Consider a curve that is tangent to a given slope field at every point and call it an *integral curve.* The following theorem connects the integral curves of the direction field f and the solutions to the ODE $\dot{x} = f(t, x)$.

Theorem 1.10. *The graphs of the solutions to the first order ODE are the integral curves of the corresponding direction field.*

Exercise 2. Prove the last theorem.

Now using this geometric interpretation I can say that to solve the IVP (1.8) it (geometrically) means to find an integral curve belonging to the corresponding direction field, which passes through the point (t_0, x_0). Theorem 1.6 now can be restated in the following way: Through each point $(t_0, x_0) \in U$, where U is the domain in which f is continuous in t and continuously differentiable in x, passes one and only one integral curve.

Example 1.11. Consider the ODE

$$\dot{x} = t^2 - x.$$

Its direction field is shown in Figure 1. The green curves represent the integral curves, i.e., the graphs of the solution to the ODE. Can you figure out how you can draw the direction field without, say, a computer?

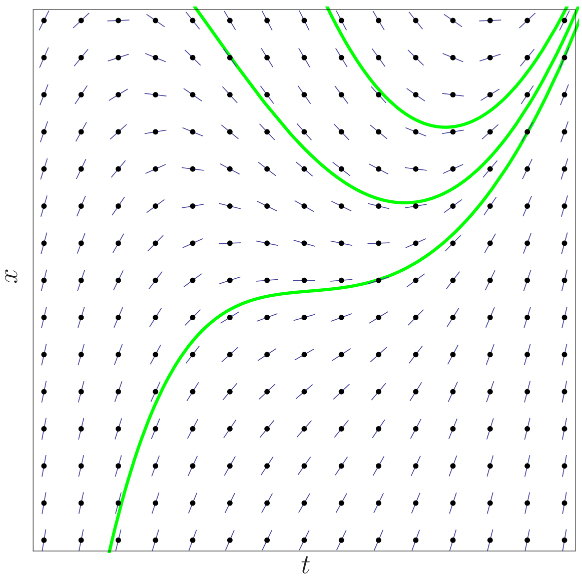

Figure 1: The direction field of $f(t,x) = t^2 - x$. The green curves are the integral curves, i.e., the graphs of the solutions to $\dot{x} = t^2 - x$.